On the Relationship between Period and Cohort Mortality*

John R. Wilmoth
Department of Demography
University of California, Berkeley

Paper for the Annual Meeting of the
Population Association of America,
Session on Statistical Demography

Philadelphia, March 31 – April 2, 2005

ABSTRACT

In this paper I explore the formal relationship between period and cohort mortality, focusing on a comparison of measures of mean life span. I consider not only the usual measures (life expectancy at birth for time periods and birth cohorts) but also some alternative measures that have been proposed recently.

I examine (and reject) the claim made by Bongaarts and Feeney that the *level* of period e_0 is distorted, or biased, due to changes in the timing of mortality. I show that their proposed alternative measure, called "tempo-adjusted" life expectancy, is exactly equivalent in its generalized form to a measure proposed by both Brouard and Guillot, the cross-sectional average length of life (or *CAL*), which substitutes cohort survival probabilities for their period counterparts in the calculation of mean life span. I conclude that this measure does not in any sense correct for a distortion in period life expectancy at birth, but rather offers an alternative measure of mean life span that is approximately equal to two analytically interesting quantities: 1) the mean age at death in a given year for a hypothetical population subject to observed historical mortality conditions but with a constant annual number of births; and 2) the mean age at death, λ, for a cohort born λ years ago.

However, I also observe that the *trend* in period e_0 does indeed offer a biased depiction of the pace of change in mean life span from cohort to cohort. Holding other factors constant, an historical increase in life expectancy at birth is somewhat faster when viewed from the perspective of cohorts (i.e., year of birth) than from the perspective of periods (i.e., year of death).

* The title differs from what appears in the program for this meeting because the content originally intended for this paper was broadened and then split into two parts. A companion paper, "On the relationship between period and cohort fertility," is in preparation.

1. Introduction

A classic problem in formal demography is how to define summary measures of demographic events for time periods that correspond in some meaningful way to the lived experience of actual cohorts. Although such period measures may not be equivalent to the analogous measure for any particular cohort, they should nevertheless represent the lifetime experience of a hypothetical cohort that is subject throughout its life to currently observed demographic conditions. The question, of course, is how to define the concept of current conditions, especially when such conditions are changing. For example, several authors have pointed out that under certain conditions the standard measure of lifetime completed fertility, the total fertility rate (*TFR*), misrepresents the average number of births that a woman would bear over her lifetime (Hajnal, 1947; Ryder, 1964; Bongaarts and Feeney, 1998). Since the problem is caused by changes from year to year in the timing of fertility as a function of age, this phenomenon is now commonly referred to as tempo distortion, or bias.

In the case of fertility, the existence of such a distortion is widely acknowledged, even though there are substantial differences of opinion about how best to adjust the *TFR* to remove such bias (Schoen, 2004; Wilmoth, 2005). In the case of mortality, however, the recent claim by Bongaarts and Feeney (2002, 2003) of a similar bias affecting period life expectancy at birth, e_0, has not found wide acceptance. Without doubt, such skepticism derives in part from the dissimilarity of the two examples, since the *TFR* measures the *number* of births over the life course, whereas e_0 depicts the *average age* at death. This difference recalls Ryder's emphasis on the fundamental distinction between the quantum and the tempo of demographic events (Ryder, 1978).

The recent discussion of these topics has revealed a pressing need to clarify the meaning of various summary measures of average longevity in a population. Therefore, in this paper I explore the formal relationship between period and cohort mortality, with a particular emphasis on the concept of mean life span. I consider not only the usual measures (life expectancy at birth for periods and cohorts) but also some alternative measures that have been proposed recently.

I examine (and reject) the assertion that the *level* of period e_0 is distorted, or biased, due to changes in the timing of mortality. I show that the alternative measure proposed by Bongaarts and Feeney, called "tempo-adjusted" life expectancy, is exactly equivalent in its generalized form to a measure proposed by both Brouard (1986) and Guillot (2003), known as the cross-sectional average length of life (or *CAL*), which substitutes cohort probabilities of survival for their period counterparts in the calculation of mean life span. I conclude that this measure does not in any sense correct for a distortion in period life expectancy at birth, but rather offers an alternative measure of mean life span that is approximately equal to two analytically interesting quantities: 1) the mean age at death in a given year for a hypothetical population subject to observed historical mortality conditions but with a constant annual number of births; and 2) the mean age at death, λ, for a cohort born λ years ago.

However, I also observe that the *trend* in period e_0 does indeed offer a biased depiction of the pace of change in mean life span from cohort to cohort. Holding other factors constant, an historical increase in life expectancy at birth is somewhat faster when viewed from the perspective of cohorts (i.e., year of birth) than from the perspective of periods (i.e., year of death).

2. Overview and Fundamental Concepts

Demographic events mark major life course transitions (e.g., birth, marriage, fertility, migration, retirement, widowhood, death). Their likelihood of occurrence within some time interval is often described using rates (and/or conditional probabilities), whose specificity may vary as a function of age, time, sex, and other individual characteristics. Such rates are often used to calculate a variety of summary measures that depict the intensity and/or timing of such events over the life course. Without doubt, the two most common of these measures are life expectancy at birth, e_0, and the total fertility rate (*TFR*).

An overview of demographic summary measures must begin with certain fundamental concepts, including three important dichotomies: (a) cohorts vs. period; (b) quantum vs. tempo; and (c) population dynamics vs. synthetic cohorts. In addition to these three distinctions, we need to understand the phenomenon of partial (or excess) quantum, which affects the period *TFR* (and all measures of quantum) whenever the timing of fertility (or other event) is changing over time. To address these and other issues in this paper, I describe a new class of models that can be used to explore mortality (and other demographic) trends based on simple assumptions about changes in the age distribution of events, rather than the age pattern of risk.

2.1 Cohorts vs. periods

Cohorts and periods are two different ways of reckoning time used for the analysis of demographic events. A *cohort* is an actual group of persons who experience a major life event around the same time. For example, birth cohorts are composed of individuals who are born in the same year. Cohort life expectancy at birth is the observed average age at death for this group (ignoring migration). In the same context, a *period* is a time interval (e.g., year, decade) and is associated with a synthetic cohort, which is an imaginary group of people who experience, hypothetically, the demographic conditions of that period throughout life. Thus, period life expectancy at birth is the expected average age at death for a synthetic cohort that experiences the mortality risks of that time (as reflected in age-specific death rates) from birth onward.

2.2 Quantum vs. tempo

In general, *quantum* refers to the intensity (or level, or frequency) with which some demographic event occurs in a population. Quantum can be described as a function of age (e.g., age-specific rates) or summarized over the entire life course (e.g., the lifetime count or probability of an event). Age-specific measures of quantum always have the number of events in the numerator. In the case of mortality, these include death counts, probabilities of death or survival, and death rates. In contrast, *tempo* refers to the timing of a demographic event over the life course. Measures of tempo are expressed in units of time (or age) and usually depict the duration until an event's occurrence. The most common example is life expectancy at birth, but other measures of mortality tempo include percentiles of the distribution of age at death (e.g., median age at death) and person-years of survival (within some interval of time and/or age).

2.3 Population dynamics vs. synthetic cohorts

There are two classes of period measures used for summarizing the demographic events of a given time interval: (a) those that describe population dynamics, and (b) those that depict the hypothetical experience of a synthetic cohort. These two types of measures serve different purposes, and a measure that is appropriate in one case may be inappropriate in the other. For example, the period total fertility rate (*TFR*), which equals the sum of observed age-specific fertility rates for a given period, depicts accurately the average contribution to population change attributable to the current fertility of woman in the reproductive age range (Calot, 2001).

However, as a measure of lifetime fertility for a synthetic cohort, the *TFR* has at least two inherent flaws. First, as discussed in the following section, it is affected by the phenomenon of partial (or excess) quantum whenever there are changes in the timing of fertility as a function of age. This problem, often called "tempo distortion" or "bias," can be circumvented by a small adjustment applied to age-specific fertility rates, which has the effect of replacing (or removing) the partial (or excess) quantum caused by changes in fertility tempo. Second, observed age-specific fertility rates reflect past as well as current fertility patterns, since they depend on the distribution of women by parity at each age. This problem can be avoided by computing an alternative measure of period total fertility based on parity transition rates within a multi-state framework (Wilmoth, 2005).

Thus, even though it is usually presented as a measure of lifetime fertility for a synthetic cohort, it is more appropriate to interpret the *TFR* as a measure of population dynamics.[1] If we desire a measure of total fertility that depicts the lifetime experience of a synthetic cohort based only on current fertility conditions, then we must address both of the problems mentioned above. These arguments are elaborated in a companion paper, which makes the case for replacing the traditional *TFR* by a pair of period measures: (a) the net reproduction rate (*NRR*) for the analysis of population dynamics, and (b) a full-quantum (or tempo-adjusted) multi-state *TFR* to represent the lifetime reproduction of a synthetic cohort.

In the case of mortality as well, some measures of mean life span are useful mostly for the analysis of population dynamics. For example, the cross-sectional average length of life (*CAL*) depicts the size and age distribution of a population at a point in time given its past mortality trends but assuming (hypothetically) a constant annual stream of births (Guillot, 2003). As shown here, *CAL* is also approximately equal to certain measures of mean life span for the population in question. For example, it is quite similar in form to the mean age at death that would be observed in a given time period for a population with identical mortality patterns and a constant number of births per year. In both these cases, however, *CAL* is describing population dynamics, not the life course of a synthetic cohort based exclusively on the mortality conditions of the given period. That purpose is fulfilled uniquely by the period life expectancy at birth, e_0, which gives the expected mean age at death implied by the observed death rates of that time.

[1] The *TFR* is often interpreted (at least implicitly) as a proxy for the net reproduction rate (*NRR*). For example, since population replacement in low-mortality populations requires a *TFR* of about 2.1 children per woman, a convenient approximation in such situations is that $NRR \approx TFR/2.1$.

In general, measures of the average age of an event over the life course have these two common forms: (a) *observed* mean age in the population itself assuming a constant birth stream; and (b) *expected* mean age in a synthetic cohort assuming that current age-specific transition rates are experienced over a lifetime. Some confusion results from the fact that different traditions have existed in fertility and mortality analysis concerning the appropriate definition for the mean age of the event. Perhaps because a central focus of fertility studies has been the role of reproduction in population dynamics, the definition of "average age at birth" has followed the concept of an *observed* mean age. In contrast, it was quite sensible for life expectancy at birth to reflect the concept of an *expected* mean age, since mortality studies have been framed in terms of risk reduction and abstract notions of quality of life, not population dynamics.[2]

2.4 Causes and consequences of partial (or excess) quantum

Many demographic events, like death, occur at various ages for members of the same cohort. An associated *probability distribution* depicts the timing of such events as a function of age, and thus also in relation to the time periods in which they occur. During a given time period, each living cohort undergoes some fraction of its total lifetime experience of the event in question, and the total number of events observed during that period is a composite of these fractional segments of cohort lifetimes.

If the age distribution of events is identical from cohort to cohort, a period cross-section of these fractional segments sums to one, and therefore the collection of events within the period can be said to represent the equivalent of a complete cohort lifetime. However, whenever there are changes in the distribution of events by age for successive cohorts, a period cross-section of cohort probability distributions typically does not sum to one, and thus period events generally misrepresent the equivalent of a complete cohort lifetime.[3] Thus, a delay in the timing of events from cohort to cohort produces a phenomenon of *partial quantum*, whereas an acceleration of timing results in *excess quantum* during the period in question. (To simplify the exposition here, I will often consider only the case of tempo delay and partial quantum, since the causes and consequences of excess quantum are identical, though always in the opposite direction.)

The phenomenon of partial (or excess) quantum is the source of a "tempo distortion," or "bias," that affects measures of lifetime quantum, like the *TFR*. This distortion can be easily eliminated by adjusting age-specific fertility rates in an appropriate fashion (Wilmoth, 2005). However, as noted earlier, this distortion is relevant only in situations where the *TFR* is interpreted as measure of lifetime fertility for a synthetic cohort. When the *TFR* is employed as a measure of population dynamics, the partial (or excess) quantum caused by changes in fertility tempo is a desirable outcome. Adjusting the measure in such cases to remove tempo effects creates a bias where none existed before.

The role of these factors in the analysis of quantum measures, like the *TFR*, is relatively straightforward, owing to the fact that the model of a synthetic cohort is relatively simple in that case. In order to represent the lifetime quantum of an event, such as total fertility, demographers

[2] Admittedly, the use of "observed" mean age is not entirely correct in this context, since I am referring to a hypothetical situation in which births are assumed to be constant over time. However, for lack of a better idea about how to label this crucial distinction (between "observed" and "expected" mean ages), I have retained this terminology for the time being.
[3] A sum of one in this case could occur only by coincidence, if negative and positive factors cancelled out, but such an occurrence seems extremely unlikely.

have typically created a synthetic cohort that is not subject to mortality or other forms of attrition, and thus the base population that accumulates events (e.g., births) over the life course is constant. For this reason, adjusting for the effects of partial quantum (or tempo delay) is a simple matter of replacing the fraction of events for each cohort that have been postponed from the time period in question into the future.

In contrast, tempo measures and their associated synthetic cohorts have a more complicated mathematical structure due to the phenomenon of attrition, which affects the base population (e.g., number of survivors) that is eligible to experience a given event (e.g., death). In such cases, adjusting for tempo delay (or partial quantum) has a dual effect. For a given base population, it restores a fraction of events that have been postponed into the future. However, it also alters the base population itself at each age. Whereas the first effect has a relatively minor effect on measures of mean age (e.g., life expectancy at birth), the latter effect is quite significant and fundamentally alters the nature of the measure. In fact, as I show here, tempo adjustment has the effect of converting a period survival probability (i.e., the probability of survival to age x within a period life table) into an analogous cohort survival probability (i.e., the probability of survival to age x for the cohort born x years ago). In doing so, it converts period e_0 into *CAL*, and thus fundamentally alters the nature of the measure (recall the earlier discussion of synthetic cohorts vs. population dynamics).

In short, adjusting for tempo change in the case of a tempo measure has the effect of removing historical changes in the quantity being measured. Tempo adjustment in this case converts a period measure based on a synthetic cohort into a cross-sectional measure that reflects the past experiences of cohorts. As noted earlier, the primary use for *CAL* is the analysis of population dynamics. Differences between *CAL* and period e_0 do not suggest that the latter measure is “distorted” in any sense. Rather, the two measures differ because they describe different things.

2.5 Models of mortality change over time

[This section not finished yet]

This project uses a new class of models to gain insights into period-cohort relationships. Previously, most models of mortality change over age and time have been specified in terms of trends in death rates. Here, changes in mortality are specified in terms of shifting distributions of deaths by age. Rate models vs. percentile models.

3. Mortality Functions and Basic Relationships

3.1 Single Cohort Model

For a single cohort (real or synthetic), the usual formulas for computing life expectancy at birth are the following:

$$\begin{aligned} e_0 &= \int_0^\infty x\,\phi(x)\,dx \\ &= \int_0^\infty x\,\ell(x)\,\mu(x)\,dx \ , \\ &= \int_0^\infty \ell(x)\,dx \end{aligned} \tag{1}$$

where $\phi(x) = -\frac{d}{dx}\,\ell(x) = \ell(x)\,\mu(x)$ is the probability density function, describing the distribution of deaths by age in the cohort; $\mu(x) = -\frac{d}{dx}\,\ell(x)\big/\ell(x) = -\frac{d}{dx}\ln\ell(x)$ is the death rate at age *x*; $H(x) = \int_0^x \mu(a)\,da$ is the cumulative death rate at age *x*; and $\ell(x) = e^{-H(x)} = e^{-\int_0^x \mu(a)\,da} = \int_x^\infty \phi(a)\,da$ is the probability of survival from birth until exact age *x*.

Although they have different forms, all three of the above formulas yield the same value for the mean age at death in a cohort. The difference between the first two formulas is trivial, since $\phi(x) = \ell(x)\,\mu(x)$. Both of these formulas depict life expectancy at birth as an average age at death, or as an expected value associated with the probability distribution. However, the last formula is different in both form and substance; it suggests an alternative interpretation of mean life span as the average accumulation of person-years lived by members of the cohort.

It is also possible to depict life expectancy at birth as an integral with respect to the probability of dying. Such calculations are closely related to percentiles of the distribution, $\tilde{a}(\pi)$, which are defined as follows:

$$\tilde{a}(\pi) = x \quad \text{such that} \quad \pi = \Phi(x) = 1 - \ell(x) \ , \tag{2}$$

where $\Phi(x) = \int_0^x \phi(a)\,da$ is the distribution (or cumulative probability) function for ages at death in the cohort. Thus, $\tilde{a}(\pi)$ is an age, *x*, such that the proportion of total deaths (over the cohort's lifetime) occurring before age *x* is π. Thus, the derivative of π with respect to age, *x*, equals the density function at that age:

$$\tfrac{d}{dx}\pi = \phi(x) \ . \tag{3}$$

Substituting $\tilde{a}(\pi)$ in place of *x*, the relationships described in equation 2 can also be written as follows:

$$\pi = \Phi(\tilde{a}(\pi)) \quad \text{and} \quad \tfrac{d}{dx}\pi = \phi(\tilde{a}(\pi)) \, . \tag{4}$$

Moreover, substituting $x = \tilde{a}(\pi)$ and/or $d\pi = \phi(x)\,dx$ in equation 1 yields the following alternative forms for life expectancy at birth:

$$\begin{aligned} e_0 &= \int_0^1 \tilde{a}(\pi)\ d\pi \\ &= \int_0^1 \frac{1}{\mu(\tilde{a}(\pi))} d\pi \end{aligned} \quad . \tag{5}$$

Thus, if we assign equal weight to arbitrarily small intervals of age, each containing an equal share of the lifetime probability of death (totaling one, of course), then life expectancy at birth equals the (weighted) average of either the mean age or the reciprocal of the death rate within each interval.

3.2 Standard Period-Cohort Model

The above formulas describe the calculation of life expectancy at birth for just one cohort, which could be either an actual birth cohort or a synthetic cohort derived from the collective mortality experience of cohorts alive during some time period. Using long series of historical data (mostly from vital statistics and census data), a common problem is to construct series of annual life tables for both periods and cohorts. To accomplish this goal, it is necessary to make some assumption about the link between period and cohort mortality, so that the two sets of tables are related in some logical and consistent manner.

The traditional manner of defining this link has been to equate period and cohort mortality in terms of their age-specific death rates. Thus, we typically begin by assuming that the death rates for a period life table should be derived directly from observed cohort death rates. In continuous age and time, this relationship can be expressed as follows:

$$\mu(x,t) = \mu_p(x,t) = \mu_c(x,t-x) = \mu_c(x,\tau) \ , \tag{6}$$

where $\tau = t - x$. Thus, by definition, the period death rate at age *x* and time *t*, $\mu(x,t) = \mu_p(x,t)$, equals $\mu_c(x,t-x) = \mu_c(x,\tau)$, the death rate at age *x* for the cohort born *x* years ago at time τ. Given this assumption, a complete series of historical life tables for both periods and cohorts is fully defined by the surface of age-specific rates expressed as functions of age and time.[4]

For example, life expectancy at birth for a given period *t* can be computed using the above equations. Written using a complete notation, the standard equations for period life expectancy at birth are as follows:

[4] The equations given here refer to the death rate at a point of age and time, (x,t), which simplifies the task of defining the link between period and cohort mortality. In practice, period and cohort mortality must be defined and measured over some time interval, such as a single calendar year. In such situations, one simple approach is merely to equate period rates to cohort rates, or vice versa, without further manipulation. However, the rates that result from such a procedure are less precise in terms of their temporal specificity than what is obtained by constructing different sets of overlapping rates for periods and cohorts. Although derived from the same data, accurate mortality rates for periods and cohorts over discrete intervals are estimated by altering slightly the configuration of age and time used to organize the raw data (annual death counts, and estimates of exposure-to-risk in person-years), so that each set of rates corresponds to exact period or cohort age intervals. For purposes of the present discussion, such complications can safely be ignored, since the mathematical development pursued here is expressed entirely in terms of continuous age and time.

$$\begin{aligned} e_0^p(t) &= \int_0^\infty x\,\phi_p(x,t)\,dx \\ &= \int_0^\infty x\,\ell_p(x,t)\,\mu_p(x,t)\,dx \ , \\ &= \int_0^\infty \ell_p(x,t)\,dx \end{aligned} \tag{7}$$

where $\phi_p(x,t) = -\frac{d}{dx}\ell_p(x,t) = \ell_p(x,t)\,\mu_p(x,t)$ gives the probability distribution of ages at death for the synthetic cohort of period *t*; $\mu_p(x,t) = \mu(x,t) = -\frac{d}{dx}\ell_p(x,t) \big/ \ell_p(x,t) = -\frac{d}{dx}\ln \ell_p(x,t)$ is the death rate at age *x* and time *t*; $H_p(x,t) = \int_0^x \mu_p(a,t)\,da$ is the cumulative rate function at age *x* and time *t*; and $\ell_p(x,t) = e^{-H_p(x,t)} = e^{-\int_0^x \mu_p(a,t)\,da} = \int_x^\infty \phi_p(a,t)\,da$ is the period probability of survival from birth until exact age *x*.

Similarly, life expectancy at birth for a cohort born at time τ can be computed as follows:

$$\begin{aligned} e_0^c(\tau) &= \int_0^\infty x\,\phi_c(x,\tau)\,dx \\ &= \int_0^\infty x\,\ell_c(x,\tau)\,\mu_c(x,\tau)\,dx \ , \\ &= \int_0^\infty \ell_c(x,\tau)\,dx \end{aligned} \tag{8}$$

where $\phi_c(x,\tau) = -\frac{d}{dx}\ell_c(x,\tau) = \ell_c(x,\tau)\,\mu_c(x,\tau)$ gives the probability distribution of ages at death for the cohort born at time τ; $\mu_c(x,\tau) = \mu(x,\tau+x) = -\frac{d}{dx}\ell_c(x,\tau) \big/ \ell_c(x,\tau) = -\frac{d}{dx}\ln \ell_c(x,\tau)$ is the cohort death rate at age *x*; $H_c(x,\tau) = \int_0^x \mu_c(a,\tau)\,da$ is the cumulative rate function at age *x*; and $\ell_c(x,\tau) = e^{-H_c(x,\tau)} = e^{-\int_0^x \mu_c(a,\tau)\,da} = \int_x^\infty \phi_c(a,\tau)\,da$ is the cohort probability of survival from birth until exact age *x*.

3.3 Cohort Distributions of Age at Death

Let us also define percentiles of the distribution of age at death for each cohort as follows:

$$\tilde{a}_c(\pi,\tau) = x \quad \text{such that} \quad \pi = \Phi_c(x,\tau) = 1 - \ell_c(x,\tau) \ , \tag{9}$$

where $\Phi_c(x,\tau) = \int_0^x \phi_c(a,\tau)\,da$ is the distribution (or cumulative probability) function for age at death in the cohort born at time τ. An important quantity in this discussion will be speed of change in these percentiles over time. Define $s_c(x,\tau)$ to be the pace of change (from cohort to cohort) in the percentile of ages at death observed at age *x* for the cohort born at time τ. Thus, by definition

$$s_c(x,\tau) = \tfrac{d}{d\tau}\tilde{a}_c(\pi,\tau) \ , \tag{10}$$

where $\pi = \Phi_c(x,\tau)$ is fixed.

In general, such quantities, known as "cohort percentile slopes," are useful for describing the relationship between period and cohort mortality. It is shown in the Appendix that a cohort percentile slope has the following relationship to the other mortality functions described above:

$$s_c(x,\tau) = \frac{-\frac{d}{d\tau}\Phi_c(x,\tau)}{\phi_c(x,\tau)} = \frac{\frac{d}{d\tau}\ell_c(x,\tau)}{\phi_c(x,\tau)} = \frac{\frac{d}{d\tau}\ln\ell_c(x,\tau)}{\mu_c(x,\tau)} = \frac{-\frac{d}{d\tau}H_c(x,\tau)}{\mu_c(x,\tau)} \ . \tag{11}$$

Thus, the cohort percentile slope at age x equals the ratio (either positive or negative) of some measure of cumulative mortality or survival, divided by an associated measure of age-specific mortality.

Using the first relationship of equation 11, it is possible to derive simple expressions for the derivatives of $\Phi_c(x,\tau)$ in three directions:

$$\tfrac{d}{d\tau}\Phi_c(x,\tau) = -s_c(x,\tau)\,\phi_c(x,\tau) \qquad \text{(horizontal)} \tag{12}$$

$$\tfrac{d}{dx}\Phi_c(x,\tau) = \phi_c(x,\tau) \qquad \text{(diagonal)} \tag{13}$$

$$\tfrac{d}{dx}\Phi_c(x,t-x) = \big(1+s_c(x,t-x)\big)\,\phi_c(x,t-x) \qquad \text{(vertical)} \tag{14}$$

As illustrated here in Figure 1, the labels, "horizontal," "diagonal," and "vertical," refer to directions of change in a Lexis diagram drawn with time (of death) along the x-axis and age along the y-axis (thus, cohort lifetimes are represented by diagonal lines). The horizontal and diagonal derivatives are obtained from the earlier equation for the percentile slope and from the definition of $\Phi_c(x,\tau)$ in terms of $\phi_c(x,\tau)$. The vertical derivative follows from the fact that the derivative in the diagonal direction equals the sum of the other two derivatives.

The derivative of $\Phi_c(x,\tau)$ in the vertical direction is important because it illustrates that the cross-sectional sum of cohort probability distributions does not in general equal one. For example, if $s_c(x,t-x) > 0$ for all x, it follows that

$$\int_0^\infty \phi_c(x,t-x)\,dx < \int_0^\infty \big(1+s_c(x,t-x)\big)\phi_c(x,t-x)\,dx = 1 \ . \tag{15}$$

In this example, since cohort percentile slopes at time t are positive, the timing of death is being delayed or postponed for each successive cohort. This equation illustrates the phenomenon of *partial quantum*, which occurs whenever the age distribution of events (deaths) is shifting upward over time. Conversely, if the distribution of deaths is shifting uniformly toward younger ages (thus, the timing of death is being advanced or accelerated), then $s_c(x,t-x)$ would be negative for all x at time t, and the above sum would be greater than unity (i.e., *excess quantum*). Let us refer to $\phi_c(x,t-x)$ as a *cross-sectional cohort probability density function*.

Assuming that $s_c(x,t-x) > -1$ for all x and t,[5] it is possible to define the following probability density function:

[5] It is theoretically possible for cohort percentiles to have slopes that are less that -1, and their reality has been confirmed by empirical observation. For expediency, this somewhat unusual situation will be not covered in this version of the present paper, as we assume that $s_c(x,t-x) > -1$. Note, however, that all formulas given here remain correct even when cohort percentile slopes dip below -1. Thus, it is merely the interpretation of the quantities, $\phi^*(x,t)$ and $\mu^*(x,t)$, as adjusted density functions and adjusted death rates that depends on this assumption.

$$\phi^*(x,t) = \left(1+s_c(x,t-x)\right)\phi_c(x,t-x) = \frac{\phi_c(x,t-x)}{1-r_c(x,t-x)} \quad , \tag{16}$$

where $r_c(x,t-x) = \dfrac{s_c(x,t-x)}{1+s_c(x,t-x)}$, and thus $1+s_c(x,t-x) = \left(1-r_c(x,t-x)\right)^{-1}$. This function sums to one over the full age range (see equation 15 above), since $s_c(x,t-x)\phi_c(x,t-x)$ replaces the missing quantum at age *x*, assuming $s_c(x,t-x) > 0$.[6] Thus, $\phi^*(x,t)$ is an *adjusted* cross-sectional cohort probability density function.

One important feature of the adjusted function, $\phi^*(x,t)$, is its relationship to the cross-sectional cohort cumulative probability and survival functions, $\Phi_c(x,t-x)$ and $\ell_c(x,t-x)$, respectively. In the following equation, note that the first integral derives from the definition of Φ_c, whereas the second integral follows from equation 14:

$$\Phi_c(x,t-x) = \int_0^x \phi_c(a,t-x)da = \int_0^x \phi^*(a,t)\, da \ . \tag{17}$$

Likewise, it follows that:

$$\ell_c(x,t-x) = 1-\Phi_c(x,t-x) = \int_x^\infty \phi_c(a,t-x)da = \int_x^\infty \phi^*(a,t)\, da \ . \tag{18}$$

The relationships linking Φ_c and ℓ_c on the one hand, to ϕ_c and ϕ^* on the other are illustrated here in Figure 2.

Following a similar logic, let us define adjusted death rates as follows:.

$$\mu^*(x,t) = \left(1+s_c(x,t-x)\right)\mu(x,t) = \frac{\mu(x,t)}{1-r_c(x,t-x)} \ . \tag{19}$$

The cumulative death rate at age *x* and time *t* also has two equivalent forms:

$$H_c(x,t-x) = \int_0^x \mu_c(a,t-x)\,da = \int_0^x \mu^*(a,t)\,da \ . \tag{20}$$

Therefore, the cohort survival probability at age *x* and time *t* can be computed using either set of death rates:

$$\ell_c(x,t-x) = \exp\left\{-\int_0^x \mu_c(a,t-x)\,da\right\} = \exp\left\{-\int_0^x \mu^*(a,t)\,da\right\} . \tag{21}$$

3.4 Alternative Measures of Period Mean Life Span

Both Brouard (1986) and Guillot (2003) have proposed the "cross-sectional average length of life," a measure known by its acronym, *CAL*, which I will also call e_0^*. By definition,

$$e_0^*(t) = CAL(t) = \int_0^\infty \ell_c(x,t-x)\,dx = \int_0^\infty \exp\left\{-\int_0^x \mu^*(a,t)\,da\right\}dx = \int_0^\infty x\,\phi^*(x,t)\,dx \ . \tag{22}$$

[6] In this discussion we will generally consider the example of cohort percentiles that increase over time, reflecting an increase in longevity. It should be evident in (almost) all cases that a decrease over time is also possible and is associated with opposite effects.

where $\mu^*(x,t)$ and $\phi^*(x,t)$ are defined as before.

Bongaarts and Feeney (2002, 2003) have proposed a "tempo-adjusted" value of life expectancy at birth, which I call e_0'. By definition,

$$e_0'(t) = \int_0^\infty \exp\left\{-\int_0^x \frac{\mu(a,t)}{1-\bar{r}_c(t)}da\right\}dx = \int_0^\infty x\,\frac{\phi_c(x,t-x)}{1-\bar{r}_c(t)}\,dx \quad , \tag{23}$$

where $\bar{r}_c(t) = \int_0^\infty r_c(x,t-x)\,\phi^*(x,t)\,dx$ is the (weighted) average value of $r_c(x,t-x)$.[7]

The two quantities, $\dfrac{\mu(a,t)}{1-\bar{r}_c(t)}$ and $\dfrac{\phi_c(x,t-x)}{1-\bar{r}_c(t)}$, are known as "tempo-adjusted" mortality functions. However, as noted also by Feeney (2004), their general form should involve an adjustment factor at each age, $1+s_c(x,t-x) = \left(1-r_c(x,t-x)\right)^{-1}$, reflecting the shift in the cohort distribution of deaths observed at that exact age, rather than some average value. For comparison, note the factor of $\left(1-\bar{r}_c(t)\right)^{-1}$ in the definition of $e_0'(t)$. Thus, *CAL* equals the generalized form of "tempo-adjusted" life expectancy at birth. Introducing a factor of $1+s_c(x,t-x)$ or $\left(1-r_c(x,t-x)\right)^{-1}$ replaces the lost quantum at age *x* that results from delay in the timing of mortality from cohort to cohort.

The close relationship between e_0^*, or CAL, and e_0', can be illustrated by re-writing equation 23 as follows:

$$e_0'(t) = \int_0^\infty x\,\frac{1-r_c(x,t-x)}{1-\bar{r}_c(t)}\,\phi^*(x,t)\,dx \quad . \tag{24}$$

So long as $r_c(x,t-x)$ does not vary widely as a function of age for a given *t*, then the ratio of $1-r_c(x,t-x)$ to $1-\bar{r}_c(t)$ will be close to one. Therefore, it is plausible that $e_0^*(t) \approx e_0'(t)$ in many situations. However, Guillot (2003) notes that the difference between *CAL* and e_0' can be substantial: for French males the difference was 2.51 years in 2001 and was even larger in earlier decades (9.24 years in 1954).

In summary, from the standpoint of a "tempo-adjusted" value of life expectancy at birth, the quantity proposed by Bongaarts and Feeney, e_0', is a special case of $CAL = e_0^*$ and is exact only when the shift between successive age distributions is strictly parallel (thus, $s_c(x,t-x)$ and $r_c(x,t-x)$ are constant over age). Therefore, e_0' is an approximation of tempo-adjusted life expectancy at birth, which in its fully general form equals CAL, or e_0^*, exactly. A correct *general* interpretation of e_0' is that it equals the mean age at death that would be observed in

[7] Note that $\bar{r}_c(t)$ also equals the pace of change over time in $CAL(t)$:

$$\tfrac{d}{dt}CAL(t) = \int_0^\infty \tfrac{d}{dt}\ell_c(x,t-x)\,dx = \int_0^\infty \frac{s_c(x,t-x)}{1+s_c(x,t-x)}\phi^*(x,t)\,dx = \int_0^\infty r_c(x,t-x)\,\phi^*(x,t)\,dx = \bar{r}_c(t)\,.$$

year t for a population with a given mortality history and, hypothetically, a constant annual number of births (see further discussion below).

4. Trends in Life Expectancy at Birth by Period and Cohort

4.1 Speed of Change in Historical Trends

Figures 3A and 3B show actual and smoothed trends in period and cohort life expectancy at birth, plotted in the usual way (by year of death for period e_0 and year of birth for cohort e_0). Then, for comparison with period e_0, Figure 3C shows Swedish cohort e_0 plotted in two ways: both by year of birth, and in relation to the time when the cohort's mean age at death actually occurs. Note that the slope of the cohort trend tends to be greater than the slope of the period trend when cohort e_0 is plotted as a function of year of birth, but less when plotted according to the period in which the cohort mean age at death actually occurs.

In part, such differences are due to fluctuations over time in historical mortality trends, which affect the mean life span of periods and cohorts in complicated ways. Such factors are beyond the scope of the present work. However, in addition to the arbitrary influences of history, there exists an intrinsic difference between period and cohort trends in e_0 due to the fundamental mathematical relationship linking the age and time of death to a decedent's time of birth.

4.2 Intrinsic Difference in Period-Cohort Slopes

As before, let $\tau = t - x$. In words, $\text{cohort} = \text{period} - \text{age}$. Clearly, when x is fixed, $d\tau = dt$. However, when x is changing, $d\tau = dt - dx$. In that case,

$$\tfrac{d\tau}{dt} = 1 - \tfrac{dx}{dt} = 1 - r \quad \text{and} \quad \tfrac{dt}{d\tau} = 1 + \tfrac{dx}{d\tau} = 1 + s\,. \tag{25}$$

where $r = \tfrac{dx}{dt}$ and $s = \tfrac{dx}{d\tau}$. Therefore,

$$1 - \tfrac{dx}{dt} = \left(1 + \tfrac{dx}{d\tau}\right)^{-1} \quad \text{or} \quad 1 - r = (1 + s)^{-1}\,. \tag{26}$$

It also follows that

$$r = \frac{s}{1+s} \quad \text{and} \quad s = \frac{r}{1-r}\,. \tag{27}$$

Thus, r and s represent two different measures of the speed of change over time in some function of age. The former is a slope with respect to the timing of the event itself, whereas the latter is with respect to the timing of birth for the cohort that experiences the event.

Figure 4 offers a simple example: A trend in which some measure of tempo increases by 1 year of age over 5 years of time (thus, $r = 0.2$). However, the same increase involves only 4 cohorts (thus, $s = 0.25$).

4.3 Period-Cohort Trends in Linear Shift Model

In order to elucidate the relationship between period and cohort mortality, it is useful to simulate historical trends using a model of a shifting distribution of age at death. The shift model I will explore here has three important characteristics:

a) It is linear (i.e., the trend in each percentile of the distribution is linear over time);

b) It is continuous (i.e., the shift extends relatively far into both the past and the future); and

c) It is defined in relation to a baseline mortality distribution associated with time $t = 0$.[8]

To simplify the exposition, the linear shift model described here is specified in terms of *period* mortality at time $t = 0$. It is also possible to define such a model as a function of *cohort* mortality at time $t = 0$ (i.e., based on a cross-section of cohort mortality distributions at this moment). However, as shown in the Appendix, a *continuous* linear shift model yields identical results for those periods and cohorts whose life spans lie fully within the shift whether the model is defined in terms of period or cohort mortality.[9] Therefore, I assume here that the time scale of the shift is relatively long (say, 150 years both forward and backward from time $t = 0$).

As was done earlier for cohorts, let us define percentiles of the period distribution of age at death (i.e., for the synthetic cohort associated with period *t*) as follows:

$$\tilde{a}_p(\pi, t) = x \quad \text{such that} \quad \pi = \Phi_p(x,t) = 1 - \ell_p(x,t) \ , \tag{28}$$

where $\Phi_p(x,t) = \int_0^x \phi_p(a,t)\,da$ is the distribution (or cumulative probability) function for age at death in period *t*. Furthermore, assume that the percentile associated with the same value of π equals *y* at time $t = 0$:

$$\tilde{a}_p(\pi, 0) = y \quad \text{such that} \quad \pi = \Phi_p(y,0) = 1 - \ell_p(y,0) \ . \tag{29}$$

The relationship between these two ages, *x* and *y*, can be used to specify the form of historical changes in the age distribution of deaths.

For example, the core assumption of the *linear* shift model is that the values of *x* associated with a given *y* form a straight line, whose slope may vary as a function of age:

$$x = y + r(y)\,t \quad (\text{for } -T < t < T\,), \tag{30}$$

where $r(y)$ can take on different values as a function of age, *y*, subject to certain restrictions (see Appendix); and *T* is the duration of the shift both forward and backward from $t = 0$. In general, let us assume that *T* is sufficiently large to assure that all cohorts alive at $t = 0$ experience the shift for their entire lives.[10]

---

[8] Time $t = 0$ is chosen as the baseline for the model in order to keep the formulas as simple as possible. If one wishes to use some other year, say t_0, as the reference point for the shift, then all formulas shown here could be modified by substituting $t' = t - t_0$ in place of *t*.

[9] Note that if the model involves an abrupt change of slope in the percentiles of a mortality distribution at some moment close to the present, say $t = 0$, then there are important differences between these two approaches.

[10] If we allow for theoretically infinite life spans, *T* should be large enough to assure that a very high proportion of deaths (say, $1 - \varepsilon$, where $\varepsilon > 0$ is very small) for cohorts alive at time $t = 0$ occur during the period of the shift.

Note that $\Phi_p(x,t) = 1 - \ell_p(x,t) = \pi$ is constant for all combinations of *x* and *t* along this *percentile contour line*. Therefore, another way of describing the core assumption of a linear shift model is that

$$\Phi_p(x,t) = \Phi(y) \quad \text{or} \quad \ell_p(x,t) = \ell(y) \tag{31}$$

where $y = x - r(y)t$. Thus, $\Phi(y)$ and $\ell(y)$ depict the baseline mortality distribution and survival probabilities for the linear shift model. They are identical to the corresponding period mortality functions associated with time $t = 0$ in the simulated population (i.e., $\Phi_p(y,0) = \Phi(y)$ and $\ell_p(y,0) = \ell(y)$).

It is shown in the Appendix that in a continuous linear shift model, period life expectancy at birth has the following form:

$$e_0^p(t) = e_0 + \bar{r}t \tag{32}$$

where $e_0 = \int_0^\infty x\,\phi(x)\,dx$; $\bar{r} = \int_0^\infty r(x)\,\phi(x)\,dx$; and $\phi(x) = \frac{d}{dx}\Phi(x) = -\frac{d}{dx}\ell(x)$. In the same model, life expectancy at birth for the cohort born in year τ is as follows:

$$e_0^c(\tau) = e_0^* + \bar{s}\,\tau + \int_0^\infty x\,s(x)\,\phi^*(x)\,dx \tag{33}$$

where $e_0^* = \int_0^\infty x\,\phi^*(x)\,dx$; $\bar{s} = \int_0^\infty s(x)\,\phi^*(x)\,dx$; $s(x) = r(x)/\big(1 - r(x)\big)$; $\phi^*(x) = \ell^*(x)\,\mu^*(x)$; $\mu^*(x) = \mu(x)/\big(1 - r(x)\big)$; and $\ell^*(x) = e^{-\int_0^x \mu^*(a)\,da}$.

In the simulated population, period life expectancy at time $t = 0$ serves as the baseline value for the linear shift model, i.e., $e_0^p(0) = e_0$. Let us consider the relationship between this quantity and cohort life expectancy for two particular cohorts:

a) The cohort born at that moment, i.e., $\tau = 0$; and

b) The cohort whose average age at death occurs at time $t = 0$.

As indicated by equation 33 above, cohort life expectancy is a function of e_0^*, or *CAL*, at time $t = 0$. For the cohort born at time $\tau = 0$, this equation simplifies to the following:

$$e_0^c(0) = e_0^* + \int_0^\infty x\,s(x)\,\phi^*(x)\,dx \tag{34}$$

However, case b) is more complicated.

Obviously the cohort whose average age at death occurs at time $t = 0$ must have been born at some earlier date, say $\tau = -\lambda$, where $\lambda > 0$. Setting $e_0^c(-\lambda) = \lambda$ in equation 33 and then solving for λ yields the following formula:

$$e_0^c(-\lambda) = \lambda = \int_0^\infty x\,\frac{1 - s(x)}{1 - \bar{s}}\,\phi^*(x)\,dx \approx e_0^* \ . \tag{35}$$

Thus, $CAL(0)$ in the simulated population is the cohort mean age at death (approximately) that is attained at time $t = 0$ by a cohort born $CAL(0)$ years earlier (approximately).

4.4 Empirical Application of Linear Shift Model

The similarity between $e_0^c(-\lambda)$ in the linear shift model and the earlier formula for $e_0'(t)$ provides the motivation for yet another measure of mean life span based on mortality conditions at time *t*. By definition, let

$$e_0''(t) = \int_0^\infty x \frac{1 - s_c(x, t-x)}{1 - \bar{s}_c(t)} \phi^*(x,t)\, dx \quad . \tag{36}$$

The quantity, $e_0''(t)$, is a linear projection of the cross-sectional cohort mortality pattern at time *t*. If historical changes mimic the linear shift model exactly, then $e_0''(t) = e_0^c(t - \lambda) = \lambda$. Even when actual conditions differ from this model, $e_0''(t)$ may serve as an approximation of cohort life expectancy for the cohort whose average age at death occurs at time $t = 0$.

In Figure 5, smoothed trends in Swedish period and cohort life expectancy at birth are compared to the predictions derived from the linear shift model. The predicted values of $e_0^c(t)$ and $e_0''(t)$ for each year are derived by re-scaling the time axis in each case so that the current year is treated as $t = 0$ in the above formulas. Thus, I create a separate linear projection based on current (cohort) mortality patterns in each year, and from each of these linear shift models I derive values of cohort life expectancy for these two cohorts. Predictions match reality reasonably well. However, the purpose of these calculations is not to obtain estimates or forecasts of cohort life expectancy, but rather to provide insights into the relationship between period and cohort mortality.

Note that $CAL(t) = e_0^*(t) \approx \frac{1}{2}\left(e_0'(t) + e_0''(t)\right)$, since $\frac{1}{2}\left(\frac{1 - r_c(x,t-x)}{1 - \bar{r}_c(t)} + \frac{1 + s_c(x,t-x)}{1 + \bar{s}_c(t)}\right) \approx 1$.

In the special case where $r_c(x, t-x) = r$ and $s_c(x, t-x) = s$ for all *x*, these relationships are exact, as the three measure are identical in this situation.

5. Average Age at Death vs. Life Expectancy at Birth

[This section not finished yet]

5.1 Mean Age at Death in the Actual Population

An observed quantity; the average age at death in a population at time t:

$$\begin{aligned}\bar{a}_1(t) &= \frac{\int_0^\infty x\,D(x,t)\,dx}{\int_0^\infty D(x,t)\,dx} = \frac{\int_0^\infty x\,N(x,t)\,\mu(x,t)\,dx}{\int_0^\infty N(x,t)\,\mu(x,t)\,dx} \\ &= \frac{\int_0^\infty x\,B(t-x)\,\ell_c(x,t-x)\,\mu(x,t)\,dx}{\int_0^\infty B(t-x)\,\ell_c(x,t-x)\,\mu(x,t)\,dx} = \frac{\int_0^\infty x\,B(t-x)\,\phi_c(x,t-x)\,dx}{\int_0^\infty B(t-x)\,\phi_c(x,t-x)\,dx}\end{aligned} \tag{37}$$

5.2 Mean Age at Death in a Constant-Birth Population

A hypothetical quantity; the average age at death at time t assuming constant birth series (equals the version of "tempo-adjusted" life expectancy at birth proposed by Bongaarts and Feeney):

$$\bar{a}_2(t) = e_0'(t) = \frac{\int_0^\infty x\,B\,\phi_c(x,t-x)\,dx}{\int_0^\infty B\,\phi_c(x,t-x)\,dx} = \frac{\int_0^\infty x\,\phi_c(x,t-x)\,dx}{\int_0^\infty \phi_c(x,t-x)\,dx} = \int_0^\infty x\,\frac{\phi_c(x,t-x)}{1-\bar{r}_c}\,dx \tag{38}$$

5.3 Mean Age at Death Implied by Cross-Sectional Cohort Survival Probabilities

A hypothetical quantity; the average age at death in a synthetic cohort for which the probability of survival to age x equals the proportion of survivors in year t for the cohort born at time $t-x$:

$$\begin{aligned}\bar{a}_3(t) &= CAL(t) = e_0^*(t) = \int_0^\infty \ell_c(x,t-x)\,dx \\ &= \int_0^\infty \exp\left\{-\int_0^x \mu^*(a,t)\,da\right\}dx = \int_0^\infty x\,\phi^*(x,t)\,dx = \int_0^\infty x\,\ell_c(x,t-x)\,\mu^*(x,t)\,dx\end{aligned} \tag{39}$$

5.4 Mean Age at Death Implied by Current Death Rates

A hypothetical quantity; the average age at death in a cohort that experiences the mortality risks observed at time t, as measured by age-specific death rates, i.e., $e_0^p(t)$ (period life expectancy at birth):

$$\begin{aligned}\bar{a}_4(t) &= e_0^p(t) = \int_0^\infty \ell_p(x,t)\,dx = \int_0^\infty \exp\left\{-\int_0^x \mu(a,t)\,da\right\}dx \\ &= \int_0^\infty x\,\phi_p(x,t)\,dx = \int_0^\infty x\,\ell_p(x,t)\,\mu(x,t)\,dx\end{aligned} \tag{40}$$

5.5 Comparison of Four Measures

To compare these four formulas, it is helpful to write them using similar but abbreviated forms (omitting values in parentheses and ranges of integration). Thus:

$$\bar{a}_1 = \int x \cdot \frac{\ell_c \cdot \mu}{\int B \cdot (1 - r_c) \cdot \phi_c^* da \Big/ B} dx \tag{41}$$

$$\bar{a}_2 = \int x \cdot \ell_c \cdot \frac{\mu}{1 - \bar{r}_c} dx \approx \int x \cdot \ell_c \cdot \mu^* \, dx \tag{42}$$

$$\bar{a}_3 = \int x \cdot \ell_c \cdot \frac{\mu}{1 - r_c} dx = \int x \cdot \ell_c \cdot \mu^* \, dx \tag{43}$$

$$\bar{a}_4 = \int x \cdot \ell_p \cdot \mu \, dx \tag{44}$$

6. Conclusion

[This section not finished yet]

Trends in period life expectancy at birth misrepresent the lived experience of cohorts in terms of the *speed* of change in the average length of life.

Proposed alternative measures (*CAL* and e_0') reflect different conceptualizations of "average age at death," not a correction for a distortion in period e_0.

7. Appendix

7.1 Percentile Slopes of Cohort Distributions of Age at Death

Recall from the main text that percentiles of the distribution of age at death for the cohort born at time τ are defined as follows:

$$\tilde{a}_c(\pi, \tau) = x \quad \text{such that} \quad \pi = \Phi_c(x, \tau) = 1 - \ell_c(x, \tau) \ , \tag{9}$$

where $\Phi_c(x,\tau) = \int_0^x \phi_c(a,\tau)\, da$ is the distribution (or cumulative probability) function for age at death in the given cohort. Also recall the following definition for the pace of change (from cohort to cohort) in the percentile of this distribution that occurs at age x:

$$s_c(x, \tau) = \tfrac{d}{d\tau} \tilde{a}_c(\pi, \tau) \ , \tag{10}$$

where $\pi = \Phi_c(x,\tau)$ is fixed. I show here that this *cohort percentile slope* has the following equivalent forms:

$$s_c(x,\tau) = \frac{-\frac{d}{d\tau}\Phi_c(x,\tau)}{\phi_c(x,\tau)} = \frac{\frac{d}{d\tau}\ell_c(x,\tau)}{\phi_c(x,\tau)} = \frac{\frac{d}{d\tau}\ln \ell_c(x,\tau)}{\mu_c(x,\tau)} = \frac{-\frac{d}{d\tau}H_c(x,\tau)}{\mu_c(x,\tau)} \ . \tag{11}$$

For simplicity, let us consider a single age x for the cohort born at time τ, and let $s = s_c(x,\tau)$. As noted by Bongaarts and Feeney (2002), in order for s to equal the slope of the percentile associated with age x for the given cohort, it must satisfy the following equation:

$$\tfrac{d}{da}\Phi_c(x+sa,\tau+a)\Big|_{a=0} = 0 \quad . \tag{45}$$

That is, a change of s units in x accompanied by a unit change in τ is associated with no change whatsoever in the cumulative probability of death, π, in the immediate vicinity of x and τ. Let $y = x + sa$ and $u = \tau + a$. It follows that:

$$\tfrac{d}{da}\Phi_c(x+sa,\tau+a) = \tfrac{d}{dy}\Phi_c(y,u)\tfrac{dy}{da} + \tfrac{d}{du}\Phi_c(y,u)\tfrac{du}{da} = \phi_c(y,u)\,s + \tfrac{d}{du}\Phi_c(y,u) \quad . \tag{46}$$

Setting $a = 0$ and equating the result to zero gives us the following expression:

$$\phi_c(x,\tau)\,s + \tfrac{d}{d\tau}\Phi_c(x,\tau) = 0 \quad . \tag{47}$$

Solving for s yields the first relationship in equation 11. The other three forms of s are an immediate result of the following elementary relationships: (a) $\tfrac{d}{d\tau}\ell_c(x,\tau) = -\tfrac{d}{d\tau}\Phi_c(x,\tau)$; (b) $\tfrac{d}{d\tau}\ln \ell_c(x,\tau) = -\tfrac{d}{d\tau}H_c(x,\tau)$; and (c) $\phi_c(x,\tau) = \ell_c(x,\tau)\,\mu_c(x,\tau)$.

7.2 Fundamental Properties of Linear Shift Model

7.2.1 Period and Cohort Life Expectancy at Birth

As stated in the main text, a linear shift model can be specified by assuming that each percentile of the period distribution of age at death (i.e., within successive period life tables for the simulated population) is constant along a line defined by the following equation:

$$x = y + r(y)\,t \quad (\text{for } -T < t < T\,), \tag{30}$$

where $r(y)$ can take on different values as a function of age, y, subject to certain restrictions (see section 7.2.3 below); and T is the duration of the shift both forward and backward from $t = 0$. Note that $\tfrac{dx}{dt} = r(y)$ for all combinations of x and t along this line. In other words,

$$\Phi_p(x,t) = \Phi(y) \quad \text{or} \quad \ell_p(x,t) = \ell(y) \tag{31}$$

where $y = x - r(y)\,t$. Note that $\tfrac{dy}{dx} = 1\big/\big(1 + r'(y)\,t\big)$, where $r'(y) = \tfrac{d}{dy}r(y)$.

Thus, $\Phi(y)$ and $\ell(y)$ depict the baseline mortality distribution and survival probabilities for the linear shift model. They are identical to the corresponding period mortality functions associated with time $t = 0$ for the simulated population (i.e., $\Phi_p(y,0) = \Phi(y)$ and $\ell_p(y,0) = \ell(y)$). Differentiating $\Phi_p(x,t)$ with respect to age, we obtain the period distribution of deaths by age:

$$\phi_p(x,t) = \tfrac{d}{dx}\Phi_p(x,t) = \tfrac{d}{dy}\Phi(y)\cdot\tfrac{dy}{dx} = \frac{\phi(y)}{1 + r'(y)\,t} \quad . \tag{48}$$

Then, using the change of variable from x to y given above, we may calculate period life expectancy at birth as follows:

$$\begin{aligned} e_0^p(t) &= \int_0^\infty x\,\phi_p(x,t)\,dx \\ &= \int_0^\infty \big(y + r(y)\,t\big)\phi(y)\,dy \\ &= \int_0^\infty y\,\phi(y)\,dy + t\int_0^\infty r(y)\,\phi(y)\,dy \\ &= e_0 + \bar{r}t \end{aligned}, \tag{49}$$

where $e_0 = \int_0^\infty y\,\phi(y)\,dy$ and $\bar{r} = \int_0^\infty r(y)\,\phi(y)\,dy$.

Furthermore, the death rate at age x and time t is given by:

$$\mu(x,t) = \mu_p(x,t) = \frac{\phi_p(x,t)}{\ell_p(x,t)} = \frac{\mu(y)}{1 + r'(y)\,t}\ , \tag{50}$$

where $\mu(y) = \phi(y)/\ell(y) = -\frac{d}{dy}\ell(y)\big/\ell(y)$ is the death rate at age y implied by the baseline mortality distribution. For the cohort born at time τ, the death rate at age x is as follows:

$$\mu_c(x,\tau) = \mu(x,\tau + x) = \frac{\mu(z)}{1 + r'(z)\,(\tau + x)} \tag{51}$$

where $z = x - r(z)\,(\tau + x)$. Note that $\frac{dz}{dx} = \big(1 - r(z)\big)\big/\big(1 + r'(z)\,(\tau + x)\big)$. Also note that $x = z + s(z)\,(\tau + z)$, where $s(z) = r(z)/\big(1 - r(z)\big)$.

Now, using the change of variable from x to z given above, we can compute the cumulative death rate for the cohort born at time τ as follows:

$$H_c(a,\tau) = \int_0^a \mu_c(x,\tau)\,dx = \int_0^a \frac{\mu(z)}{1 + r'(z)\,(\tau + x)}\,dx = \int_0^b \mu^*(z)\,dz = H^*(b) \tag{52}$$

where $b = a - r(b)\,(\tau + a)$; and $\mu^*(z) = \mu(z)\big/\big(1 - r(z)\big)$. Based on this formula, it is simple to compute the probability of survival to age x for the cohort born at time τ:

$$\ell_c(x,\tau) = e^{-H_c(x,\tau)} = e^{-H^*(z)} = e^{-\int_0^z \mu^*(a)\,da} = \ell^*(z) \tag{53}$$

Therefore, the cohort distribution of deaths by age is as follows:

$$\phi_c(x,\tau) = \ell_c(x,\tau)\,\mu_c(x,\tau) = \frac{\ell^*(z)\,\mu(z)}{1 + r'(z)\,(\tau + x)} = \frac{\big(1 - r(z)\big)\phi^*(z)}{1 + r'(z)\,(\tau + x)} \tag{54}$$

where $\phi^*(z) = \ell^*(z)\,\mu^*(z)$. Then, using the same change of variable from x to z, we can calculate cohort life expectancy at birth as follows:

$$
\begin{aligned}
e_0^c(\tau) &= \int_0^\infty x\,\phi_c(x,\tau)\,dx \\
&= \int_0^\infty \big(z + s(z)(\tau+z)\big)\phi^*(z)\,dz \\
&= \int_0^\infty z\,\phi^*(z)\,dz + \tau\int_0^\infty s(z)\,\phi^*(z)\,dz + \int_0^\infty z\,s(z)\,\phi^*(z)\,dz \\
&= e_0^* + \bar{s}\,\tau + \int_0^\infty z\,s(z)\,\phi^*(z)\,dz
\end{aligned}
\tag{55}
$$

where $e_0^* = \int_0^\infty z\,\phi^*(z)\,dz$ and $\bar{s} = \int_0^\infty s(z)\,\phi^*(z)\,dz$.

7.2.2 Equivalence of Period- and Cohort-based Models

The linear shift model described in the main text and in the previous sub-section is specified in terms of period mortality. In other words, mortality change for the simulated population is described as a shift in the distribution of age at death in a series of period life tables. Thus, by design the period percentile slope at age x in year t, $r_p(x,t)$, equals $r(y)$, where $y = x - r(y)\,t$. We can confirm this relationship by differentiating the period survival probability, $\ell_p(x,t)$, with respect to time t:

$$
\tfrac{d}{dt}\ell_p(x,t) = \tfrac{d}{dy}\ell(y)\,\tfrac{dy}{dt} = \frac{\phi(y)}{1+r'(y)\,t}\,r(y) = \phi_p(x,t)\,r(y)\ , \tag{56}
$$

since $\frac{d}{dy}\ell(y) = -\phi(y)$ and $\frac{dy}{dt} = -\,r(y)/\big(1+r'(y)\,t\big)$. Then, by the same logic used earlier to compute cohort percentile slopes (see section 7.1 above), it follows that:

$$
r_p(x,t) = \frac{\frac{d}{dt}\ell_p(x,t)}{\phi_p(x,t)} = r(y)\ . \tag{57}
$$

Note that this result merely reflects the core assumption of the model.

On the other hand, values of cohort percentile slopes in this model are a consequence of the assumptions and must be derived. As before, the change of variable used for computing cohort mortality is defined by the equation, $z = x - r(z)(\tau + x)$. Differentiating the cohort survival probability, $\ell_c(x,t)$, with respect to time τ, we obtain:

$$
\tfrac{d}{d\tau}\ell_c(x,\tau) = \tfrac{d}{dz}\ell^*(z)\,\tfrac{dz}{d\tau} = \frac{\big(1-r(z)\big)\phi^*(z)}{1+r'(z)\big(\tau+x\big)}\,\frac{r(z)}{1-r(z)} = \phi_c(x,\tau)\,s(z)\ , \tag{58}
$$

since $\frac{d}{dz}\ell^*(z) = -\phi^*(z)$ and $\frac{dz}{d\tau} = -\,r(z)/\big(1+r'(z)\big(\tau+x\big)\big)$. It follows that:

$$
s_c(x,\tau) = \frac{\frac{d}{d\tau}\ell_c(x,\tau)}{\phi_c(x,\tau)} = s(z)\ . \tag{59}
$$

Thus, a key property of the linear shift model is the close relationship between the period and cohort percentile slopes:

$$s_c(x,\tau) = \frac{r_p(x,t)}{1-r_p(x,t)} \quad \text{and} \quad r_p(x,t) = \frac{s_c(x,\tau)}{1+s_c(x,\tau)} \ , \tag{60}$$

assuming $\tau = t - x$.

Turning the derivation around, we begin by specifying that the percentiles of successive cohort distributions of age at death are constant along a line defined by

$$x = z + s(z)(\tau + z) \quad (\text{for } -T < t < T\ ) . \tag{61}$$

Note that $\frac{dx}{d\tau} = s(z)$ for all combinations of *x* and τ along this line. In other words, the linear shift model is now based on the core assumption that

$$\Phi_c(x,\tau) = \Phi^*(z) \quad \text{or} \quad \ell_c(x,\tau) = \ell^*(z) \ , \tag{62}$$

where $z = x - s(z)(\tau + z)$. Since $\tau = -x$ at time $t = 0$, this assumption implies that

$$\Phi_c(z,-z) = \Phi^*(z) \quad \text{or} \quad \ell_c(z,-z) = \ell^*(z) \ . \tag{63}$$

Therefore, the functions $\Phi^*(z)$ or $\ell^*(z)$, again depict the cross-sectional cohort mortality distribution at time $t = 0$ for the simulated population. From equation 61, we obtain $\frac{dz}{dx} = 1\big/\big(1 + s(z) + s'(z)(\tau + z)\big)$.

Note that both the change of variable implied by equation 61, and the derivative of *z* with respect to *x* that follows from this equation, are equivalent to the forms given earlier as part of the original derivation of the linear shift model. To demonstrate the equivalence of the change of variable itself, add τ to both sides of equation 61 and rearrange the equation to obtain $\tau + x = \big(1 + s(z)\big)(\tau + z)$. Since $1 + s(z) = \big(1 - r(z)\big)^{-1}$, it follows that $\tau + z = \big(1 - r(z)\big)(\tau + x)$, and thus $z = x - r(z)(\tau + x)$. Equivalence of the two forms of the derivative, $\frac{dz}{dx}$, follows from the additional observation that $s'(z)\big/\big(1 + s(z)\big) = r'(z)\big/\big(1 - r(z)\big)$. In summary:

$$z = x - s(z)(\tau + z) = x - r(z)(\tau + x) \tag{64}$$

and

$$\frac{dz}{dx} = \frac{1}{1 + s(z) + s'(z)\big(\tau + z\big)} = \frac{1 - r(z)}{1 + r'(z)\big(\tau + x\big)} \ . \tag{65}$$

Differentiating $\Phi_c(x,\tau)$ with respect to age, the cohort distribution of deaths by age is as follows:

$$\phi_c(x,\tau) = \frac{d}{dx}\Phi_c(x,t) = \frac{d}{dz}\Phi^*(z)\cdot\frac{dz}{dx} = \frac{\phi^*(z)}{1 + s(z) + s'(z)\big(\tau + z\big)} \ . \tag{66}$$

Dividing through by $\ell_c(x,\tau) = \ell^*(z)$, we obtain the death rate at age *x* for the cohort born at time τ:

$$\mu_c(x,\tau) = \frac{\mu^*(z)}{1 + s(z) + s'(z)\big(\tau + z\big)} = \frac{\big(1 - r(z)\big)\mu^*(z)}{1 + r'(z)\big(\tau + x\big)} \tag{67}$$

Finally, the death rate at age *x* and time *t* is as follows:

$$\mu(x,t) = \mu_p(x,t) = \mu_c(x,t-x) = \frac{\mu(y)}{1+r'(y)t} \quad , \qquad (68)$$

where $y = x - r(y)t$; and $\mu(y) = (1 - r(y))\mu^*(y)$.

Therefore, the surface of death rates over age and time, $\mu(x,t)$, has the same form, and thus the relationship between all period and cohort mortality functions is the same, whether a linear shift model is specified in terms of period or cohort mortality. However, as noted in the main text, this equivalence pertains only to periods and cohorts whose entire life experience takes place within the shift. In other words, discontinuities at the start and end of the shift have different forms for period-based and cohort-based models. Since the purpose of this analysis is to understand the relationship between period and cohort mortality under conditions of stable change, such patterns are not considered here.

7.2.3 Necessary Restrictions on Percentile Slopes

[This section not finished yet]

8. Acknowledgements

This research was supported by a grant from the National Institute on Aging, R01 AG11552.

Notes

[Move footnotes here eventually]

References

Bongaarts, John, and Griffith Feeney (1998). "On the quantum of tempo of fertility." *Population and Development Review* 24(2): 271-291.

Bongaarts, John, and Griffith Feeney (2002). "How long do we live?" *Population and Development Review* 28(1): 13-29.

Bongaarts, John, and Griffith Feeney (2003). "Estimating mean lifetime." *Proceedings of the National Academy of Sciences* 100(23): 13127-13133.

Brouard, Nicolas (1986). "Structure et dynamique des populations. La pyramide des années à vivre, aspects nationaux et exemples régionaux." *Espaces, Populations, Sociétés* xx(xx): xx-xx.

Calot, Gérard (2001). "Mais qu'est-ce donc qu'un indicateur conjoncturel de fécondité?" *Population* 56(3): 325-327.

Chambers, John M., *et al*. (1983). *Graphical Methods for Data Analysis*. Boston: Duxbury.

Feeney, Griffith (2004). "A generalized mortality tempo adjustment formula." Workshop on Tempo Effects on Mortality, New York, November 18-19.

Guillot, Michel (2003). "The cross-sectional average length of life (CAL): A cross-sectional mortality measure that reflects the experience of cohorts." *Population Studies* 57(1): 41-54.

Hajnal, John (1947). "The analysis of birth statistics in the light of the recent international recovery of the birth-rate." *Population Studies* 1(2): 137-164.

Human Mortality Database (2004). University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany), available at www.mortality.org or www.humanmortality.de (data downloaded July 27, 2004).

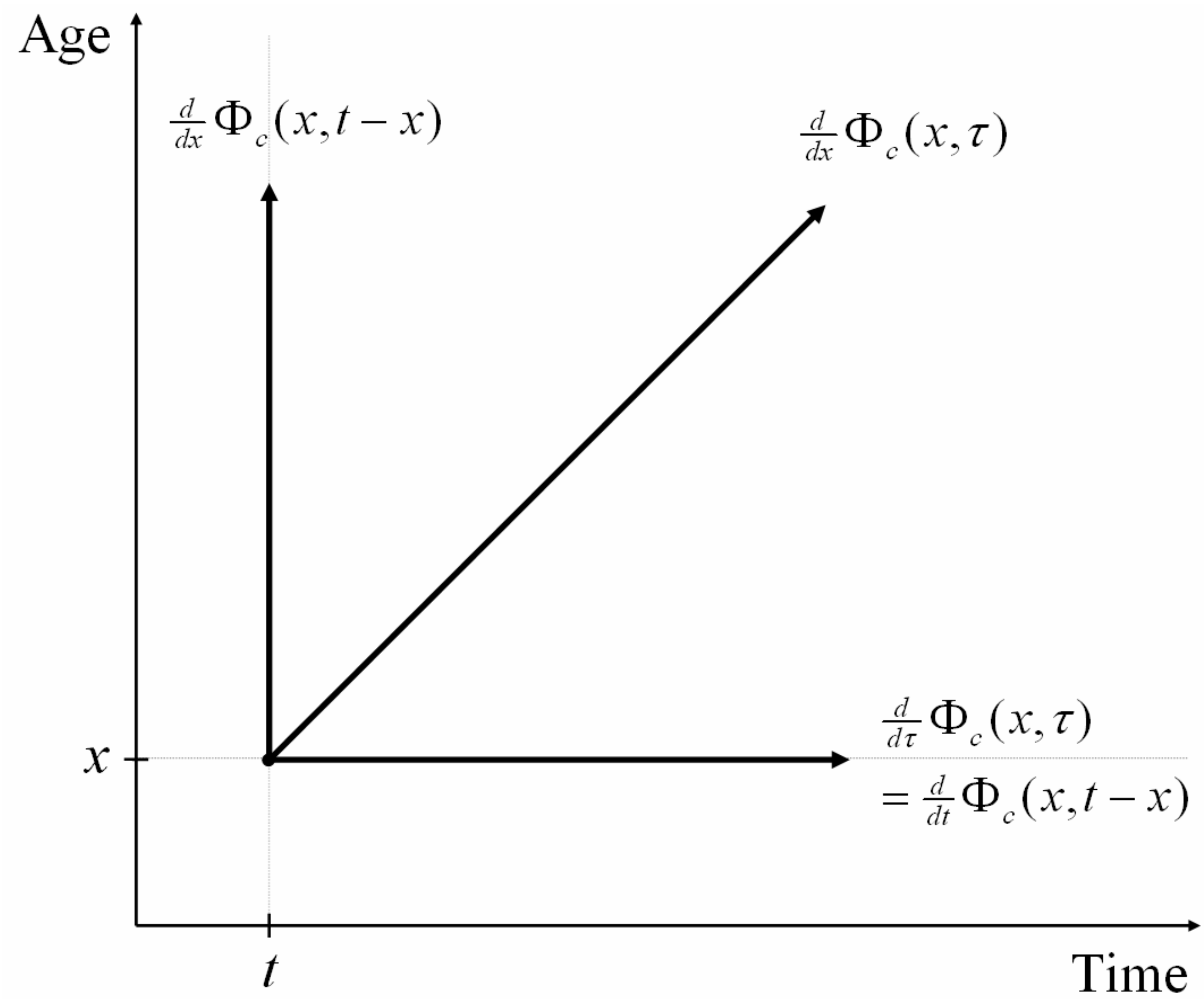
Ryder, Norman (1964). "The process of demographic translation." *Demography* 1(1): 74-82.

Ryder, Norman (1978). "A model of fertility by planning status." *Demography* 15(4): 433-458.

Schoen, Robert (2004). "Timing effects and the interpretation of period fertility." *Demography* 41(4): 801-819.

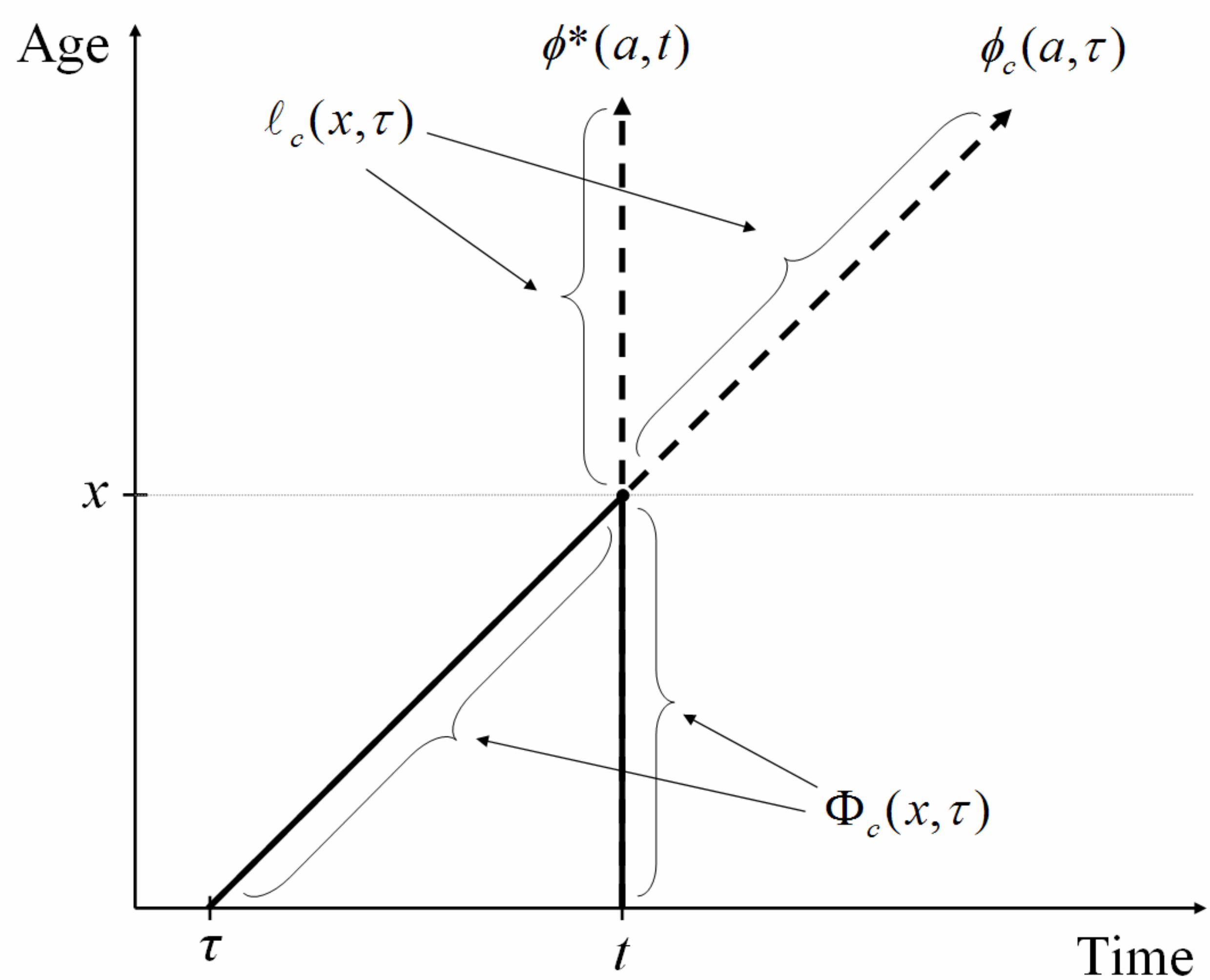
Wilmoth (2005). "On the relationship between period and cohort fertility." Manuscript (in preparation).

Figure 1
Schematic representation of derivatives in three directions
of cohort cumulative probability function, $\Phi_c(x,\tau)$



Note: By assumption, $\tau = t - x$.

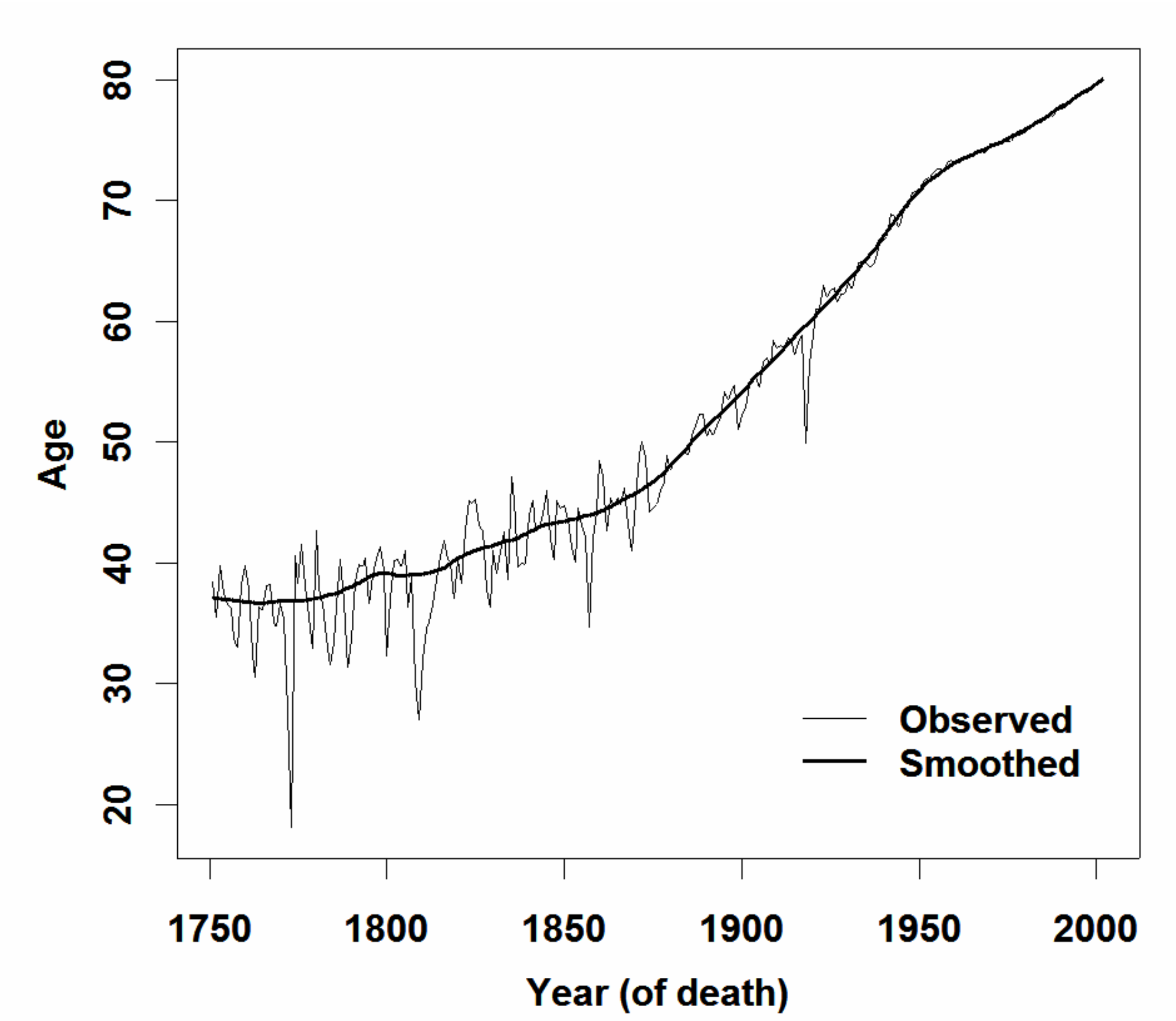
Figure 2
Two forms of cohort distribution of deaths by age, $\phi_c(a,\tau)$ and $\phi^*(a,t)$, in relation to cumulative probability of death, $\Phi_c(x,\tau)$, and probability of survival, $\ell_c(x,\tau)$



Note: By assumption, $\tau = t - x$.

Figure 3
Life expectancy at birth in Sweden

A) Periods, actual vs. smoothed trends, 1751-2002

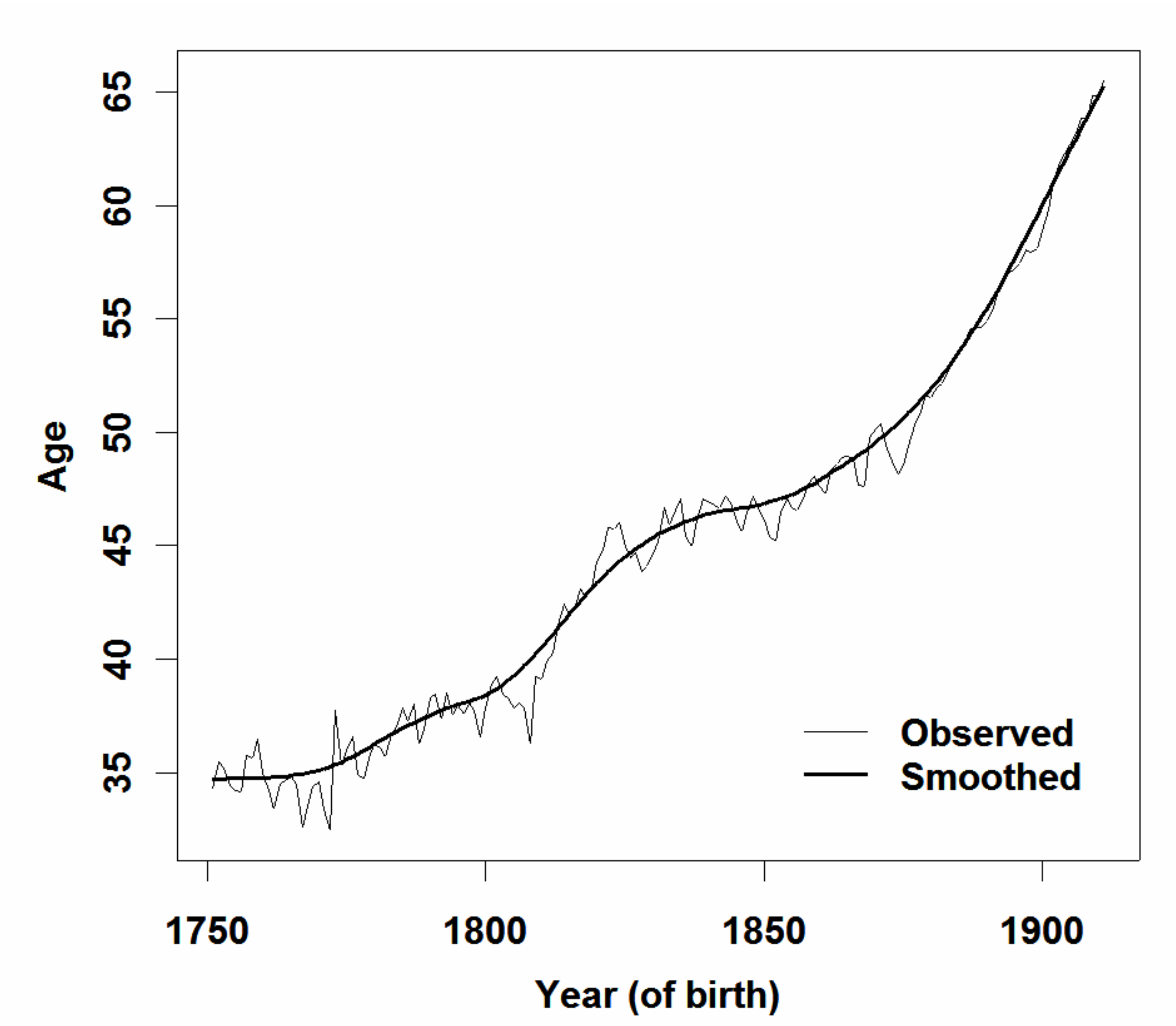


Note: The observed trend was smoothed using the LOWESS method (Chambers *et al*., 1983).

Source: Human Mortality Database (2004).

Figure 3 (cont.)

B) Cohorts, actual vs. smoothed trends, 1751-1911

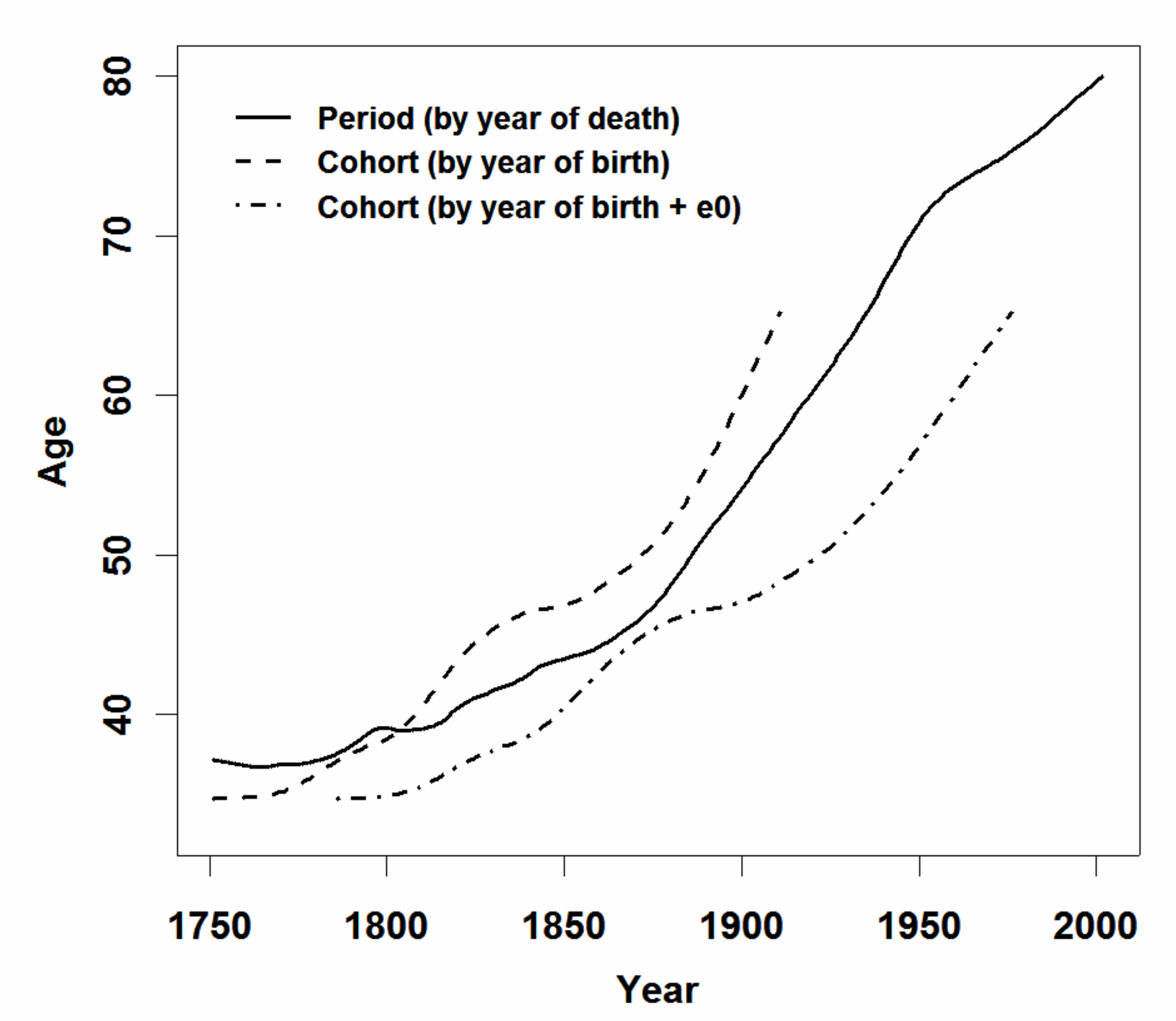


Notes: (1) See note for Figure 3A. (2) Data employed here for cohorts born after approx. 1890 are incomplete. Therefore, estimates of life expectancy at birth for these cohorts rely on recent period data at very high ages (i.e., above age 90).

Source: Human Mortality Database (2004).

Figure 3 (cont.)

C) Periods vs. cohorts, smoothed trends only



Notes: See notes for Figure 3B.

Source: Human Mortality Database (2004).

Figure 4
Simple example illustrating intrinsic difference in slope of age trend from perspective of periods ($r = 0.2$) and cohorts ($s = 0.25$)

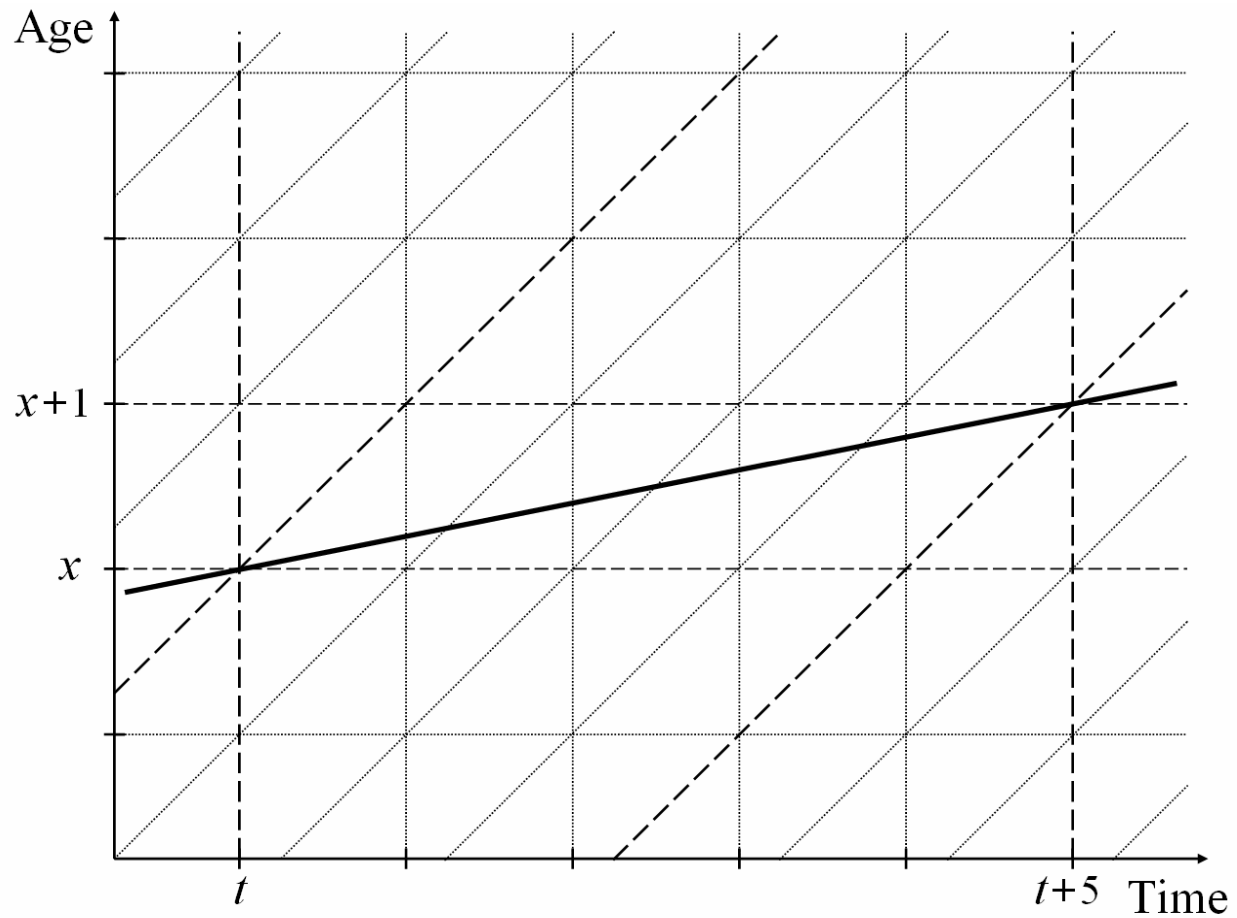


Figure 5
Life expectancy at birth in Sweden by period and cohort, plus estimates of cohort values assuming linear trends in cross-sectional cohort percentiles of age at death